Notes1 ECE 2500 Digital Logic
To the First Exam

Lecture Topics:

- The Digital World
- Boolean Algebra
- Logic Gates & Circuits
- minterms and K-maps
- Maxterms and K-maps

THE DIGITAL WORLD

Major Application of Digital Logic: the design of processor chips in computers and mobile devices.

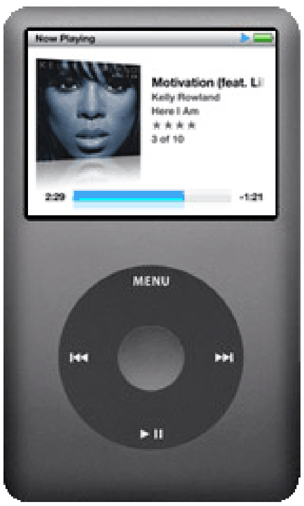
- **Classic iPod (4th generation 2004)**
  From: electronics.howstuffworks.com/ipod3.htm
  

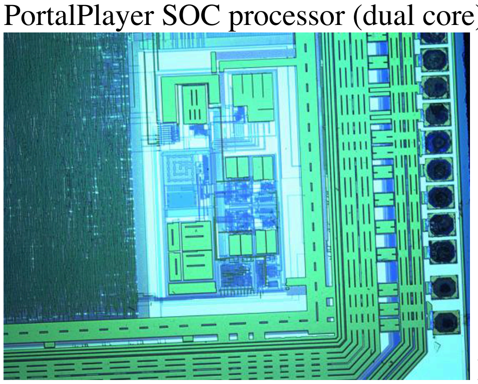
  Photo credit: apple.com
  - Display (320 x 240 pixel LCD)
    From: electronics.howstuffworks.com/lcd2.htm
  - Click Wheel (capacitive sensing controller)
    From: electronics.howstuffworks.com/ipod4.htm
  - PortalPlayer SOC processor (dual core)
    
    Photo credit: microblog.routed.net
  - Memory (SDRAM 32 MB)
  - Hard drive (30 GB)
- **iPhone** 16 Pro differences (2024)
  From: https://en.wikipedia.org/wiki/List_of_iOS_devices


Image credit: apple.com

- 6 GB memory
- 128-1012 GB solid-state drive
- Touch HDR display (2556 x 1179 pixel color OLED)
- 64 bit Apple A18 Pro SOC processor
  - Hex-core CPU
  - Hex-core GPU
  - 16-core NPU

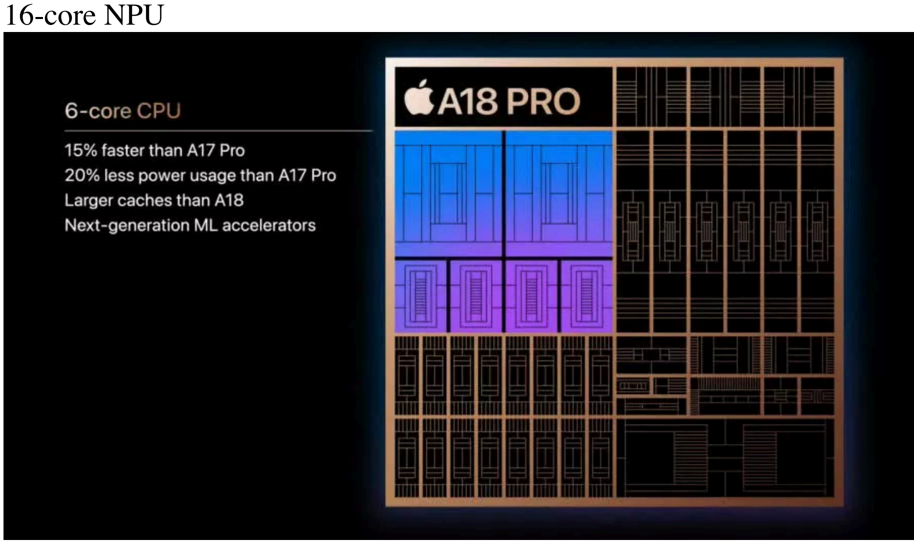
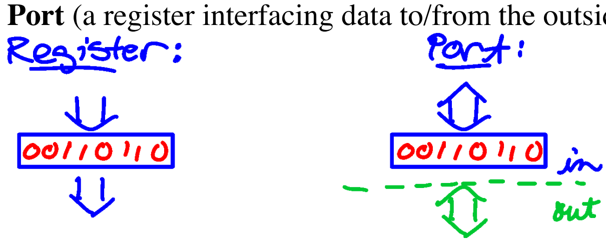


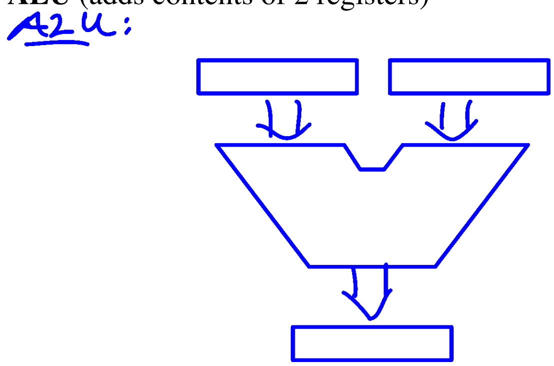
Image Credit: Apple

Digital Logic Components: these are the digital building blocks that will be studied in this course.

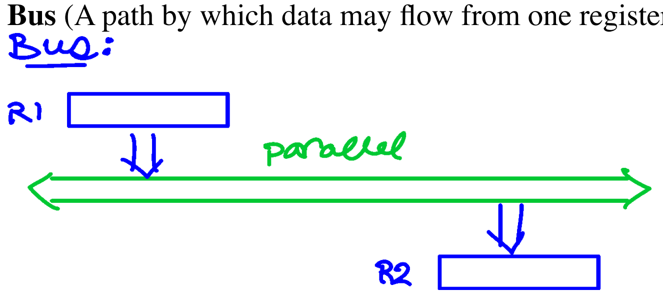
- **Register** (holds various forms of digital data)
- **Port** (a register interfacing data to/from the outside world)



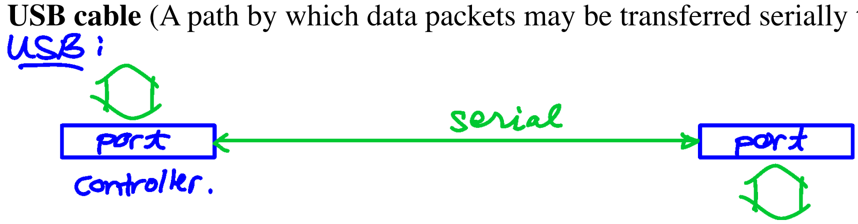
- **ALU** (adds contents of 2 registers)



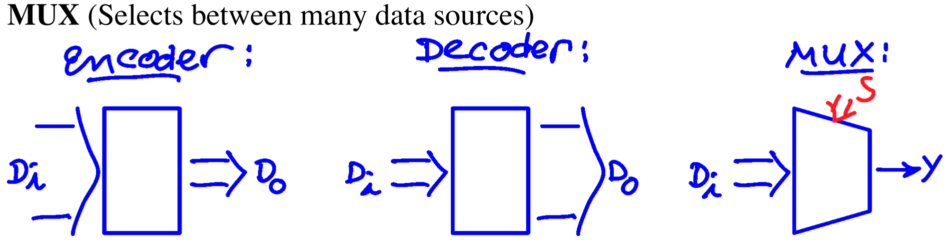
- **Bus** (A path by which data may flow from one register to another in parallel)



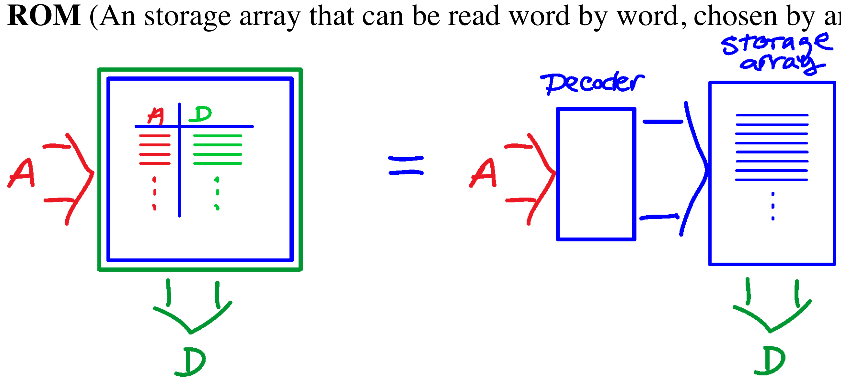
- **USB cable** (A path by which data packets may be transferred serially to ports from a hub)



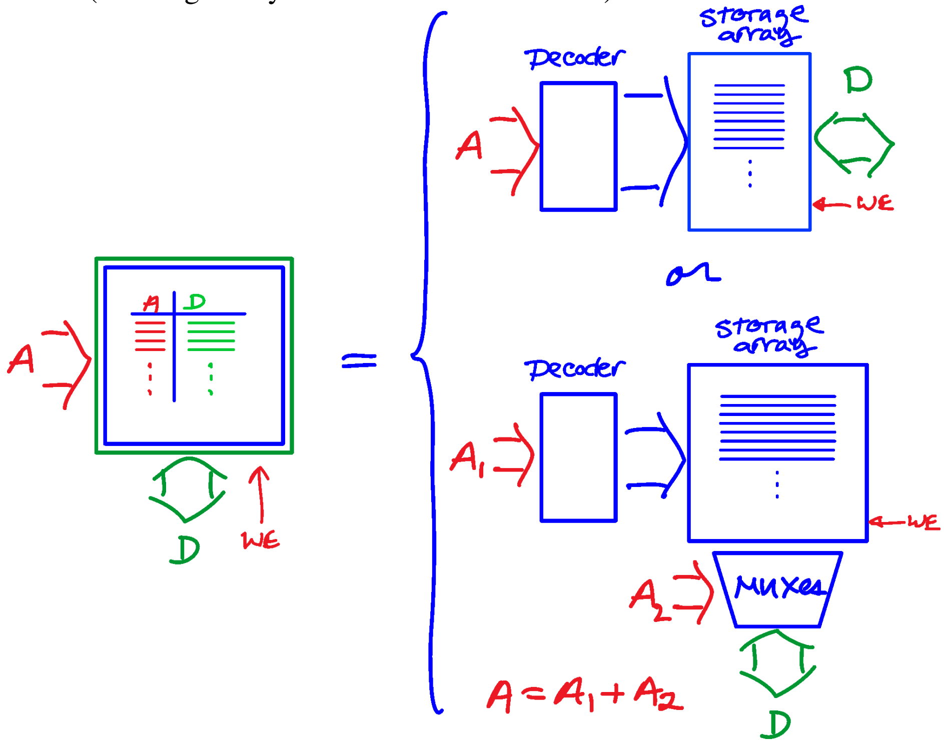
- **Encoder** (Encodes or compresses data)
- **Decoder** (Decodes or expands data. Also used to make memory location selections)
- **MUX** (Selects between many data sources)



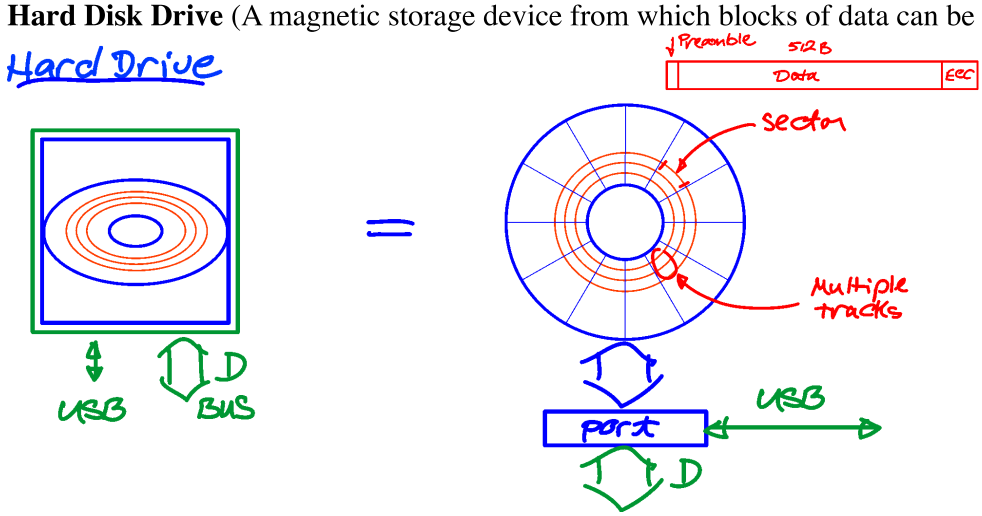
- **ROM** (An storage array that can be read word by word, chosen by an address)



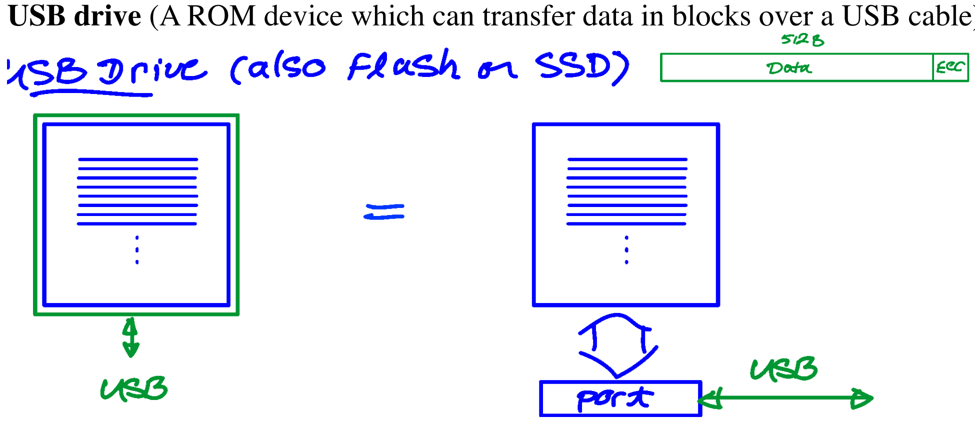
- **RAM** (A storage array that also can be written to)



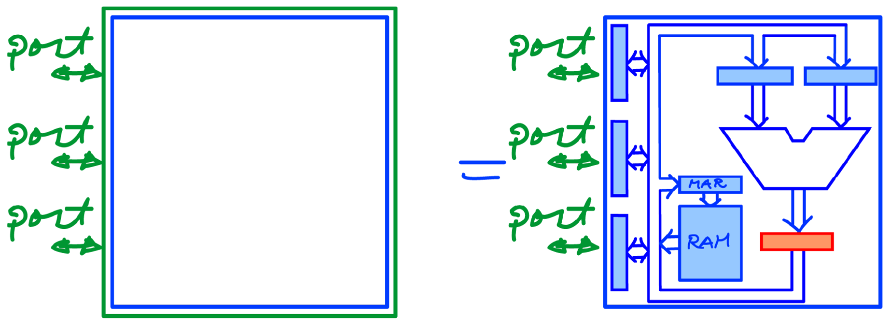
- **Hard Disk Drive** (A magnetic storage device from which blocks of data can be stored and read



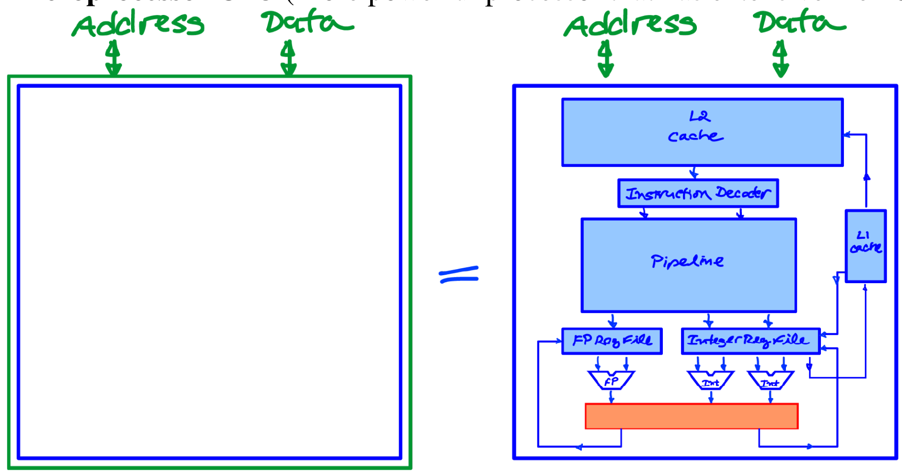
- **USB drive** (A ROM device which can transfer data in blocks over a USB cable)



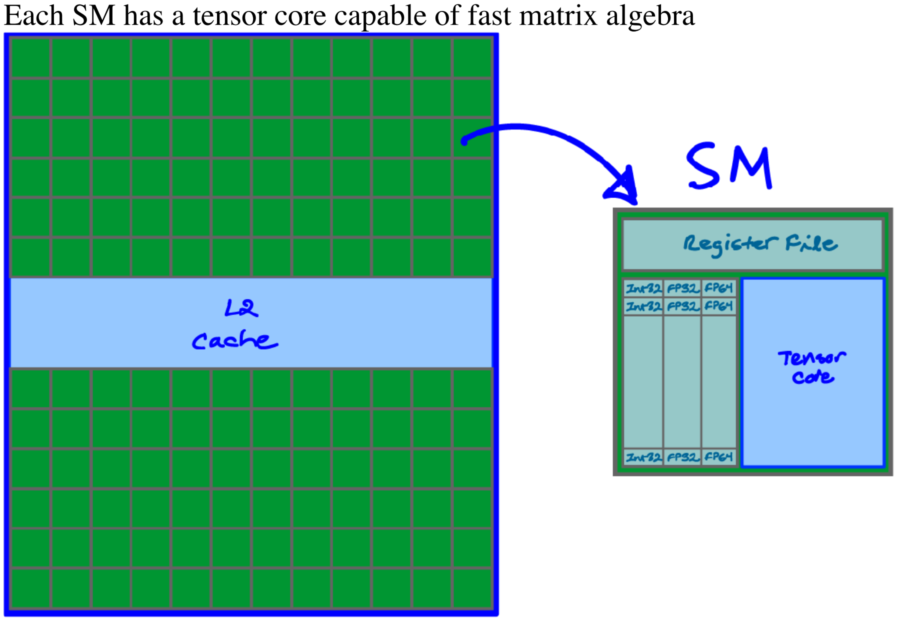
- **Microcontroller** MCU (A processing device consisting of an ALU, registers, ports and RAM)



- **Microprocessor** CPU (More powerful processor that has extensive memory and multiple ALUs)



- **Graphic Processing Unit** GPU
  - The H100 consists of 144 Streaming Multiprocessors (SM)
  - Each SM has a tensor core capable of fast matrix algebra



  - A GPT or Grot AI Large Language Model (LLM) application requires:
    - 64-256 GPUs for inference
    - 25,000 GPUs for training

Digital Data Types

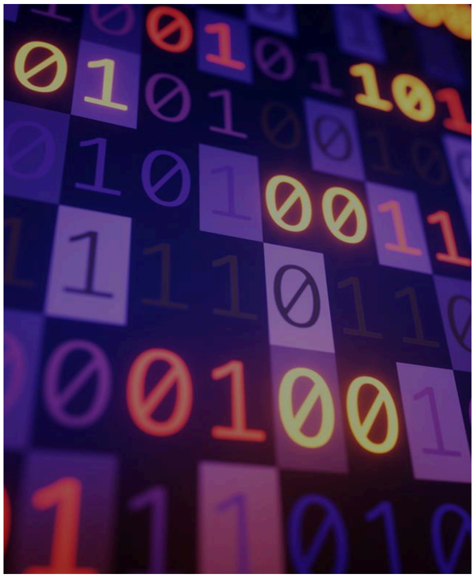
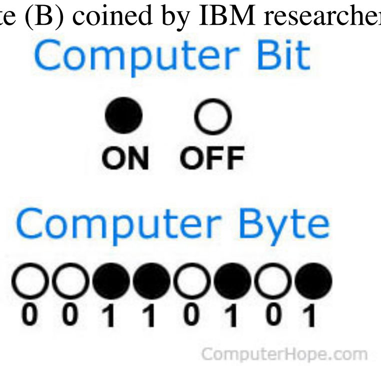
- **Numeric**

Graphic credit: techspirited.com
  - Beginnings:
    - bit (b) defined by Claude Shannon as "basic information digit" (1948)
    - byte (B) coined by IBM researcher Werner Buchholtz (1964)


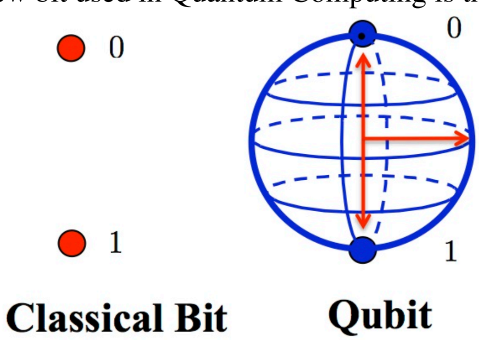
Image credit: ComputerHope.com
    - A new bit used in Quantum Computing is the Qubit:


Image credit: IBTimes UK

- Integers in a byte (8 bits)
  - Total unsigned (0 -> 255, 256 total members)

1 byte	decimal	hex
00000000	= 0	00
00000001	= 1	01
00000010	= 2	02
00000011	= 3	03
00000100	= 4	04
00000101	= 5	05
00000110	= 6	06
00000111	= 7	07

1 byte	decimal	hex
00001000	= 8	08
00001001	= 9	09
00001010	= 10	0A
00001011	= 11	0B
00001100	= 12	0C
00001101	= 13	0D
00001110	= 14	0E
00001111	= 15	0F

1 byte	decimal	hex
00010000	= 16	10
⋮	⋮	
01100011	= 99	63
01100100	= 100	64
⋮	⋮	
11001000	= 200	C8
⋮	⋮	
11111101	= 253	FD
11111110	= 254	FE
11111111	= 255	FF

  - *Example*: Hexadecimal and decimal

7 6 5 4 3 2 1 0

$\boxed{11001000} = 1\times 2^7 = 128$
$+1\times 2^6 = 64$
$+1\times 2^3 = 8$

C 8

$\boxed{0xC8 = 200}$

  - Comparison of decimal, binary, octal and hex:

Decimal	Binary	Octal	Hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F
16	10000	20	10

Red 1 and 2 = "Carry"
A,B,C,D,E,F = extra hex digits

Important number conversions to remember:

$(10)_{10} = (1010)_2 = (A)_{16}$
$(11)_{10} = (1011)_2 = (B)_{16}$

- Fractionals in a byte
  - *Example*:

1 2 3 4 5 6 7 8

$\boxed{.11001000} = 1\times 2^{-1} = .5$
$+1\times 2^{-2} = .25$
$+1\times 2^{-5} = .03125$

. C 8

$0x.C8 = .78125$

  - Integer conversions between binary, octal and hex
    - Octal: group in3 bits
    - Hex: group in 4 bits

- *Example #1:*
  Convert $(010101100)_2$ to base 8 and 16

  Add zeros → $000(010\,101\,100)_2$

  $\underbrace{010}_{2}\ \underbrace{101}_{5}\ \underbrace{100}_{4} \rightarrow (254)_8$

  $\underbrace{0000}_{0}\ \underbrace{1010}_{A}\ \underbrace{1100}_{C} \rightarrow (AC)_{16}$

- *Example #2:*
  Convert $(110.110)_2$ to base 8 and 16

  Add zeros → $0(110.110)_2 0$ ← Add zeros

  $\underbrace{110}_{6}.\underbrace{110}_{6} \rightarrow (6.6)_8$

  $\underbrace{0110}_{6}.\underbrace{1100}_{C} \rightarrow (6.C)_{16}$

  - Juxtapositional notation:
    - Integer, radix point and fraction

      $N =$ number

      $$= (\underbrace{a_{n-1}\, a_{n-2} \dots a_2\, a_1\, a_0}_{\text{Integer}}\ \underset{\text{radix point}}{.}\ \underbrace{a_{-1}\, a_{-2}\, a_{-3} \dots a_{-m}}_{\text{Fraction}})_r$$

      $a_i =$ digits
      $r =$ radix
      $0 \leq a_i < r$, always holds.

    - *Examples:* (radix = base)

      $(353.12)_{r=10}$
      $= 3 \times 10^2 +$
      $5 \times 10^1 +$
      $3 \times 10^0 +$
      $1 \times 10^{-1} +$
      2×10^{-2}

      $(1010.01)_{r=2}$
      $= 1 \times 2^3 +$
      $0 \times 2^2 +$
      $1 \times 2^1 +$
      $0 \times 2^0 +$
      $0 \times 2^{-1} +$
      1×2^{-2}

      General Polynomial:

      $N =$ number

      $$= \sum_{i=-m}^{n-1} a_i r^i$$

      $$N = N_I + N_F$$

      $= a_{n-1} r^{n-1} +$
      $a_{n-2} r^{n-2} +$
      $a_{n-3} r^{n-3} +$
      $\dots +$
      $a_2 r^2 +$
      $a_1 r^1 +$
      $a_0 r^0 +$

      $a_{-1} r^{-1} +$
      $a_{-2} r^{-2} +$
      $a_{-3} r^{-3} +$
      $\dots +$
      $a_{-m} r^{-m}$

- **Non-numeric**
  - Characters
    - ASCII: 1 B for each of $2^8 = 256$ English, control and special characters (Latin-1)

From: http://www.asciitable.com/

character   1 byte

ASCII: "A" = $\boxed{0100\ 0001}$ = 41

"B" = $\boxed{0100\ 0010}$ = 42

"C" = $\boxed{0100\ 0011}$ = 43

⋮

"ECE 2500" = 45 43 45 20 32 35 30 30

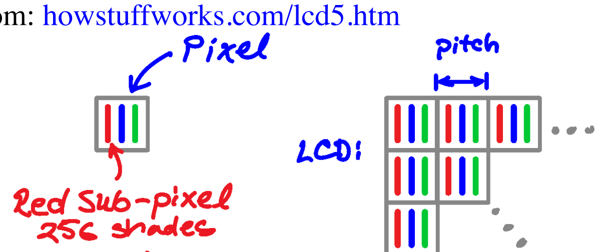
E C E sp 2 5 0 0

= 8 bytes

- UNICODE: 2^{24} ~17 million characters with code points spread over 2 or 3B (UTF-16 or 24). Handles international characters & emoticons

From: https://unicode.org/emoji/charts/full-emoji-list.html

- Color Codes
From: howstuffworks.com/lcd5.htm



R G B

RGB CODES: $\boxed{10011001}\boxed{10011001}\boxed{10011001}$ = Gray

9 9 9 9 9 9

$\boxed{11001100}\boxed{00000000}\boxed{00000000}$ = Red

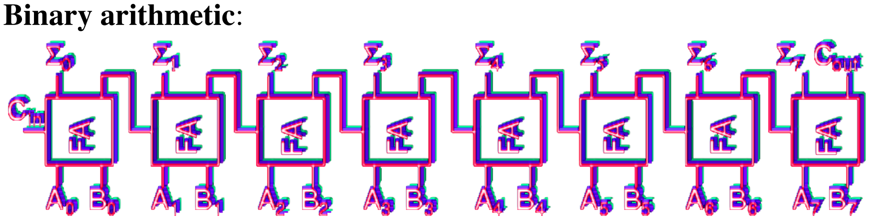
C C 0 0 0 0

$\boxed{00000000}\boxed{11111111}\boxed{00000000}$ = Green

0 0 F F 0 0

- More html examples: immigration-usa.com/html_colors.html
- 24 bit color -> 2^{24} ~ 17 million colors
  - Red 1 B => 256 shades
  - Green 1 B => 256 shades
  - Blue 1 B => 256 shades

Binary arithmetic:



- Bit by bit addition is done right to left, with **carry bits**

- *Examples:* Adding

$$\begin{array}{r} 3 \\ +4 \\ \hline 7 \end{array} \quad \begin{array}{r} 00011 \\ +00100 \\ \hline 00111 \end{array} \qquad \begin{array}{r} 5 \\ +5 \\ \hline 10 \end{array} \quad \begin{array}{r} 00101 \\ +00101 \\ \hline 01010 \end{array} \qquad \begin{array}{r} 5 \\ +7 \\ \hline 12 \end{array} \quad \begin{array}{r} 00101 \\ +00111 \\ \hline 01100 \end{array}$$

- Subtraction can be done by employing **borrow bits**, or more simply, by adding something called a 2's complement.
  - *Examples:*

$$\begin{array}{r} 10 \\ -6 \\ \hline 4 \end{array} \quad \begin{array}{r} 01010 \\ -00110 \\ \hline 00100 \end{array} \qquad \begin{array}{r} 10 \\ +(-6) \\ \hline 4 \end{array} \quad \begin{array}{r} 01010 \\ +11010 \text{ (2's C)} \\ \hline 1\,\boxed{00100} \end{array}$$

↑ overflow

- The concept of a *complement* of a decimal number:

$-N$	10's C
-1	$\rightarrow 9$
-2	$\rightarrow 8$
-3	$\rightarrow 7$
-4	$\rightarrow 6$
-5	$\rightarrow 5$
-6	$\rightarrow 4$
-7	$\rightarrow 3$
-8	$\rightarrow 2$
-9	$\rightarrow 1$

$$\begin{array}{r} 10 \\ -6 \\ \hline 4 \end{array} \rightarrow \begin{array}{r} 10 \\ +4 \\ \hline 1\,\boxed{4} \end{array} \qquad \begin{array}{r} 8 \\ -6 \\ \hline 2 \end{array} \rightarrow \begin{array}{r} 8 \\ +4 \\ \hline 1\,\boxed{2} \end{array} \qquad \begin{array}{r} 7 \\ -9 \\ \hline -2 \end{array} \rightarrow \begin{array}{r} 7 \\ +1 \\ \hline \boxed{8} \end{array} \rightarrow -2$$

- 2's complement procedure:
  - Reverse all the bits of ***N***
  - Add ***1*** to the result. This is ***N****.
  - The sign of ***N**** (as well as *N*) is shown by the most significant bit: 0 = “+”; 1 = “-“
  - *Examples:*

sign bit = '+'

$$N = 6 = \boxed{0}0110$$

$$\begin{array}{rl} 11001 & \text{step\#1} \\ +1 & \text{step\#2} \\ \hline -N = N^* = -6 = \boxed{1}1010 & \end{array}$$

sign bit = '-'

$$\text{If } N^* = \boxed{1}1110 \quad \text{what is } -N?$$

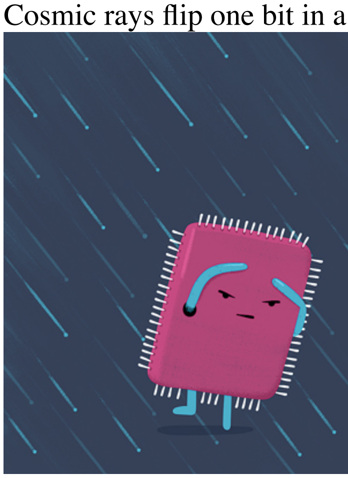
$$\begin{array}{rl} 00001 & \text{step\#1} \\ +1 & \text{step\#2} \\ \hline \boxed{0}0010 = N & \therefore \underline{-N = -2} \end{array}$$

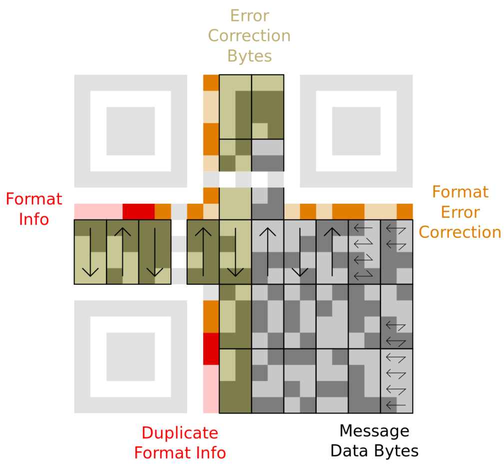
  - All the 2's complement numbers that fit into a byte.
    - 127 positive numbers *N* (sign bit = 0)
    - 128 negative numbers *N** (sign bit = 1)

- Zero (not shown)

N	decimal	$N^* = -N$	decimal
00000001	= 1	11111111	= -1
00000010	= 2	11111110	= -2
00000011	= 3	11111101	= -3
00000100	= 4	11111100	= -4
00000101	= 5	11111011	= -5
00000110	= 6	11111010	= -6
⋮	⋮	⋮	⋮
01111110	= 126	10000010	= -126
01111111	= 127	10000001	= -127
x	= 128	10000000	= -128

Error Correction Codes (ECC):

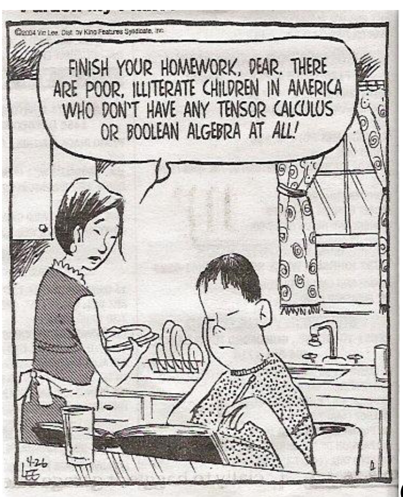
- Provides self-correction of errors that occur in the data when transporting data
  - Scratched disk alter recorded data:
     From: hardwaresecrets.com/
  - Cosmic rays flip one bit in a 4GB chip everyday:
     From: spectrum.ieee.org/

- Defaced QR code:
   From: hwww.i-programmer.info
- Reed-Solomon ECC code in QR:
  

  en.wikiversity.org/wiki/Reed%E2%80%93Solomon_codes_for_coders

Quiz #1 & selected solutions

BOOLEAN ALGEBRA



(Photo credit: Vic Lee, King Features Syndicate)

Some Preliminaries...

Binary numbers can also be used to represent truth or **logic values**.

Logic defined: the process of classifying information.

Binary logic (or more commonly, *digital logic*) is the process of classifying information into two distinct classes, e.g.

(TRUE, FALSE) = truth values
(Yes, No)
(CLOSE, OPEN) = relay positions
blown, intact = fuse state
(ON, OFF) = switch positions
(1, 0) = binary numbers, or (Logic 1, Logic 0)

Logic design is based upon the **three logic operators**

Binary Logic Operations (Variables)

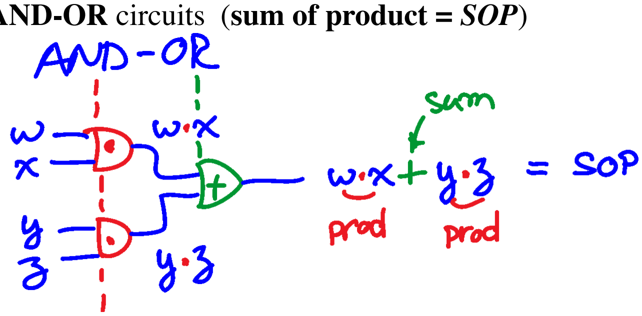
- **AND**: $z = x \bullet y$
- **OR**: $z = x+y$
- **NOT**: $z = x'$

Binary Logic Operations

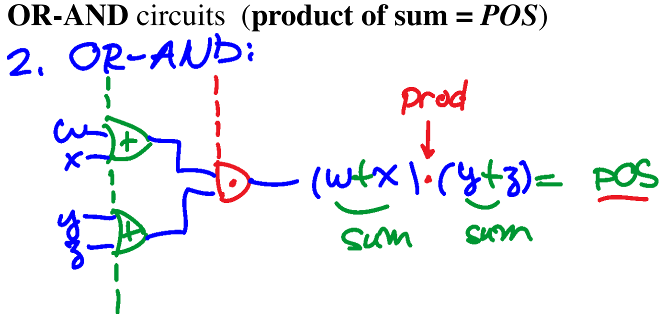
OR	XOR	AND
0 + 0 = 0	**$0 \oplus 0 = 0$**	**$0 \bullet 0 = 0$**
0 + 1 = 1	**$0 \oplus 1 = 1$**	**$0 \bullet 1 = 0$**
1 + 0 = 1	**$1 \oplus 0 = 1$**	**$1 \bullet 0 = 0$**
1 + 1 = 1	**$1 \oplus 1 = 0$**	**$1 \bullet 1 = 1$**

Two Level Logic Circuits with AND/OR/XOR gates:
From: computer.howstuffworks.com/boolean1.htm

- **AND-OR** circuits (**sum of product =** ***SOP***)



- **OR-AND** circuits (**product of sum =** ***POS***)



These circuits can also be described algebraically with the use of an algebra system for logic variables called…

Boolean Algebra

- **Fundamental properties of Boolean Algebra**: Each ***x, y*** and ***z*** are elements of ***B = {0,1}***
  1. ***Identities***: (P3, P4) (Dual)
     $x+0 = x$ $\quad$ $x \bullet 1 = x$
     $x+1 = 1$ $\quad$ $x \bullet 0 = 0$
     Also Idempotency: (P6)
     $x+x = x$ $\quad$ $x \bullet x = x$

2. ***Commutativity***: (P1)
   $x+y = y+x$ $\qquad$ $x \bullet y = y \bullet x$
3. ***Associativity***: (P2)
   $x+(y+z) = (x+y)+z$ $\qquad$ $x \bullet (y \bullet z) = (x \bullet y) \bullet z$
4. ***Distributivity***: (P8)
   $x+(y \bullet z) = (x+y) \bullet (x+z)$ $\qquad$ $x \bullet (y+z) = x \bullet y+x \bullet z$
5. ***Existence of the complement***: (P5)
   There exists an element ***x'***, called NOT ***x***, such that
   $x+x' = 1$ $\qquad$ $x \bullet x' = 0$
6. ***Involution***: (P7) $\qquad$ $(x')' = x$
7. ***Absorption***: (P12)
   $x+xy = x$ $\qquad$ $x(x+y) = x$
8. ***Adjacency***: (P9)
   $xy+xy' = x$ $\qquad$ $(x+y) \bullet (x+y') = x$
9. ***DeMorgan's Law***: (P11)
   $(x+y+z)' = x'y'z'$ $\qquad$ $(x \bullet y \bullet z)' = x'+y'+z'$

- **Duality**: Left and right hand properties above are *duals*
  - A dual may be derived by interchanging
    - 1 and 0
    - • (AND) and + (OR)
  - *Examples*:

$$x + 0 = x \qquad x \cdot (x + y) = x$$

$$x \cdot 1 = x \qquad x + xy = x$$

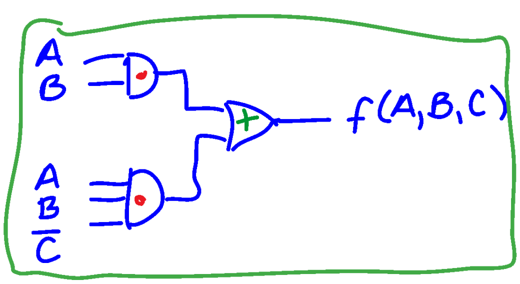
Boolean Functions and Logic Circuits

- Boolean function ***f***
  - ***f*(*A*, *B*, *C*)** is an algebraic expression of ***A*, *B*, *C***
  - ***A*, *B*, *C*** are Boolean variables
- Boolean functions are implemented by logic circuits
- Boolean functions may be simplified, resulting in simpler logic circuits
- Circuits and functions may be verified by constructing a truth table
- *Example #1:*
  - Derive a logic circuit from a Boolean function:

$$f(A,B,C) = \underbrace{A \cdot B}_{\text{products}} \overset{\text{sum}}{+} \underbrace{A \cdot B \cdot \overline{C}}_{} = \text{SOP}$$

$$= \text{sum of products}$$

Circuit 1

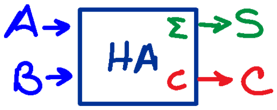


- *Example #2:*

- Derive a Boolean function for a half adder:

$$\begin{array}{r} A \\ +\,B \\ \hline C\;S \end{array} \qquad \begin{array}{r} 0 \\ +0 \\ \hline 0 \end{array} \quad \begin{array}{r} 0 \\ +1 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ +0 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ +1 \\ \hline 10 \end{array}$$

S, C

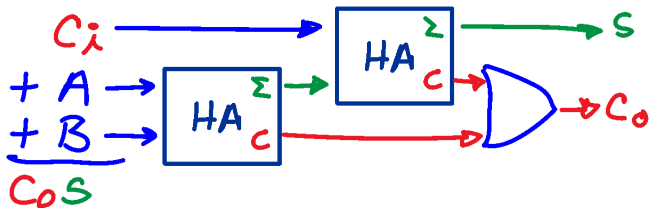


$C = AB$

$S = \bar{A}B + A\bar{B}$

$= A \oplus B$

- Make a full adder from two half adders:



* *Example#3*
  - Simplify the Boolean function of *Example#1* by pattern matching terms with the Boolean properties above:

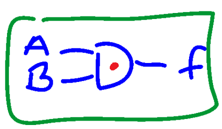
Boolean Algebra
(pattern matching)

$f = AB + AB\bar{C}$ ✓

$\underbrace{x + xy = x}_{\text{pattern}}$

Let $\begin{cases} x = AB \\ y = \bar{C} \end{cases}$

$\therefore f = \underline{\underline{AB}}$



Circuit ②

- Compare before and after circuits with a truth table:

Truth Table:

ABC	② $A \cdot B$	minterm $AB\bar{C}$	① $f = AB + AB\bar{C}$
000	0	0	0+0=0
001	0	0	0+0=0
010	0	0	0+0=0
011	0	0	0+0=0
100	0	0	0+0=0
101	0	0	0+0=0
110	1	1	1+1=1
111	1	0	1+0=1

← same →

- *Example #4:* More simplifications

$f = A\bar{B}\bar{C} + \bar{B}$

$\overbrace{A\bar{B}\bar{C}}^{y} + \bar{B}$

$\underbrace{xy + x}_{\text{pattern}} = x$

Let $\begin{cases} x = \bar{B} \\ y = A\bar{C} \end{cases}$

$\therefore f = x = \bar{B}$ ✓

$f = ABC + A\bar{B}C$

$\overbrace{ABC}^{x} + \overbrace{A\bar{B}C}^{x}$

$\underbrace{xy + x\bar{y}}_{\text{pattern}} = x$

Let $\begin{cases} x = AC \\ y = \bar{B} \end{cases}$

$\therefore f = x = AC$ ✓

- *Example #5:* DeMorgan's Law

$F = \bar{A}B + C + \bar{D}$

$\bar{F} = ?\ \overline{\underbrace{\bar{A}B}_{x} + \underbrace{C}_{y} + \underbrace{\bar{D}}_{z}}$

pattern: $\overline{x+y+z} = \bar{x}\cdot\bar{y}\cdot\bar{z}$

$\bar{F} = \overline{\bar{A}B}\cdot\bar{C}\cdot\bar{\bar{D}}$

$xy \rightarrow$ pattern: $\overline{x\cdot y} = \bar{x}+\bar{y}$

$= (\bar{\bar{A}}+\bar{B})\cdot\bar{C}\cdot D$

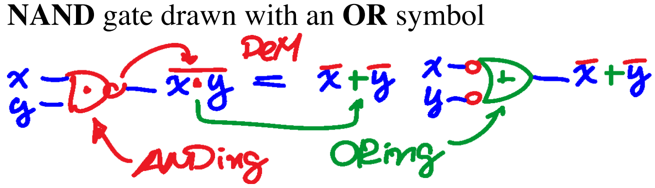
$= (A+\bar{B})\bar{C}\cdot D$

Quiz #2 & selected solutions

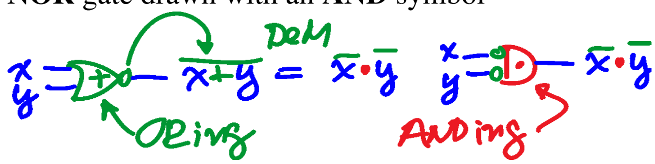
LOGIC GATES AND CIRCUITS

DeMorgan's Laws Shows Equivalent Graphical Symbols for Logic Gates: Examples are given to describe

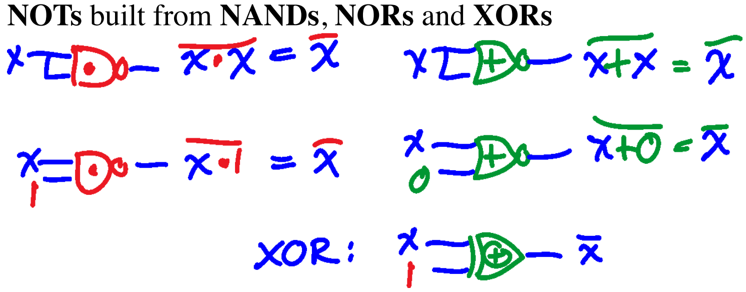
- **NAND** gate drawn with an **OR** symbol



- **NOR** gate drawn with an **AND** symbol

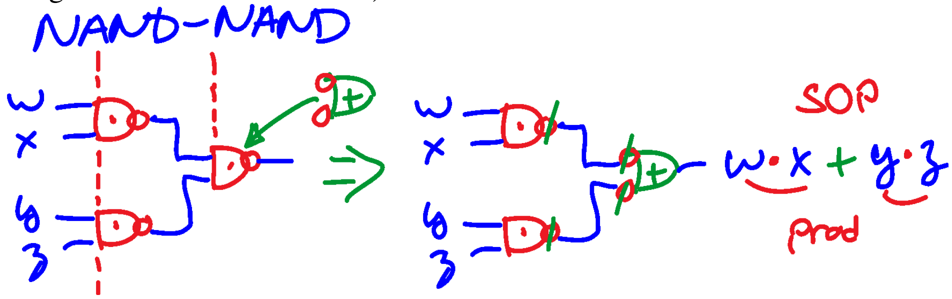


- **NOTs** built from **NANDs**, **NORs** and **XORs**

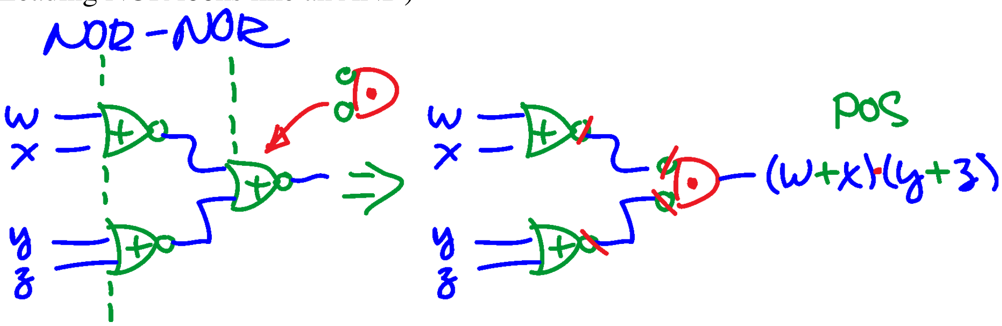


Two Level Logic Circuits with Other Gates: Examples are given to describe

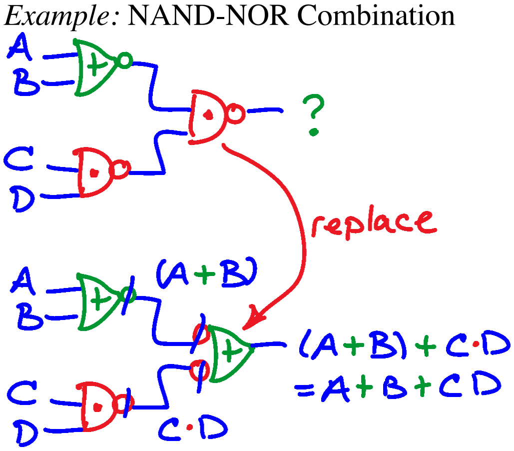
- **NAND-NAND** circuits = **AND-OR** circuits; makes ***SOP*** functions
  (Leading NAND looks like an OR)



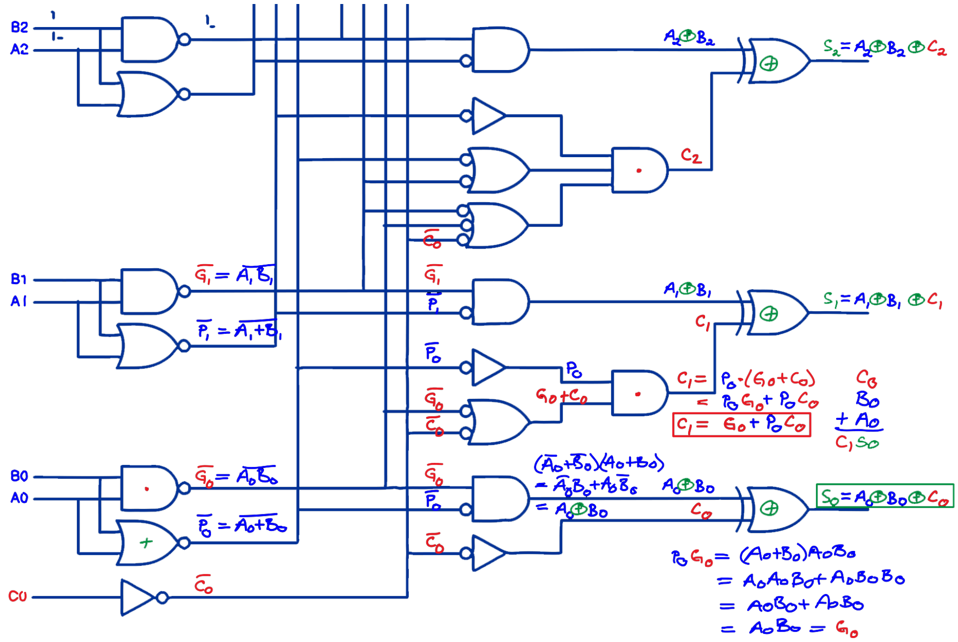
- **NOR-NOR** circuits = **OR-AND** circuits; makes ***POS*** functions
  (Leading NOR looks like an AND)



- *Example:* NAND-NOR Combination

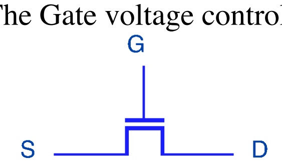


- *Example:* Carry-lookahead adder logic
  - Most heavily designed circuit in the history of electronics
  - NOT, NAND, NOR, XOR combination
  - Gate fronts and backs match so bubbles cancel



CMOS Implementation of Logic Gates
Examples are shown to implement **NAND**, **NOR**, and **NOT** gates from elementary **NMOS** and **PMOS** transistors.

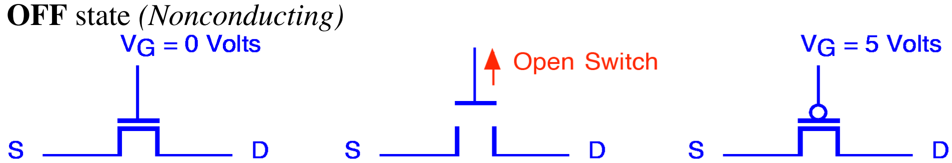
- **CMOS** transistors = NMOS plus PMOS
- Current flows between the Source and the Drain
- The Gate voltage controls the conduction value between the Source and Drain:



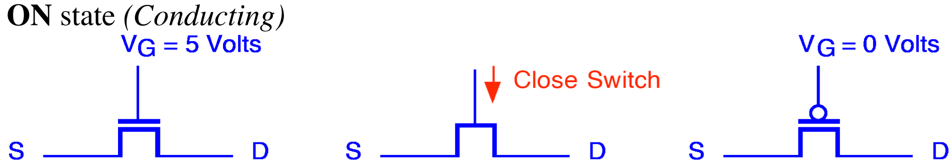
G = Gate
S = Source
D = Drain
(S and D are interchangeable)

The NMOS transistor may be configured to operate in one of three different states, as determined by the **voltage** at the gate terminal V_G:

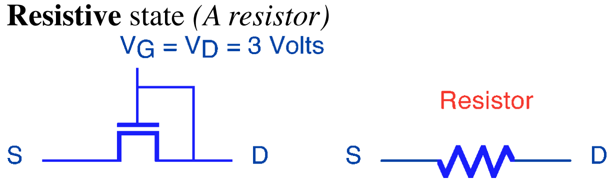
- Three states of a **NMOS** and **PMOS** transistors:
  - **OFF** state *(Nonconducting)*



  - **ON** state *(Conducting)*

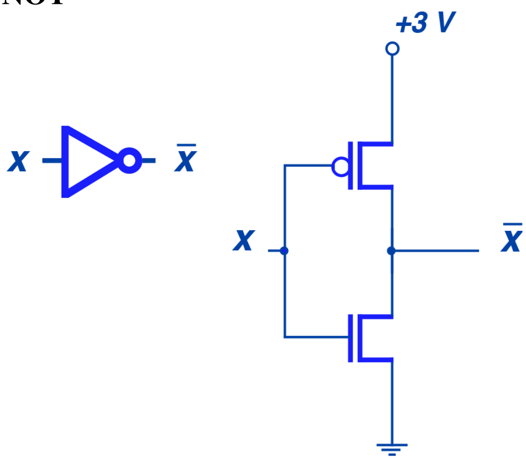


  - **Resistive** state *(A resistor)*

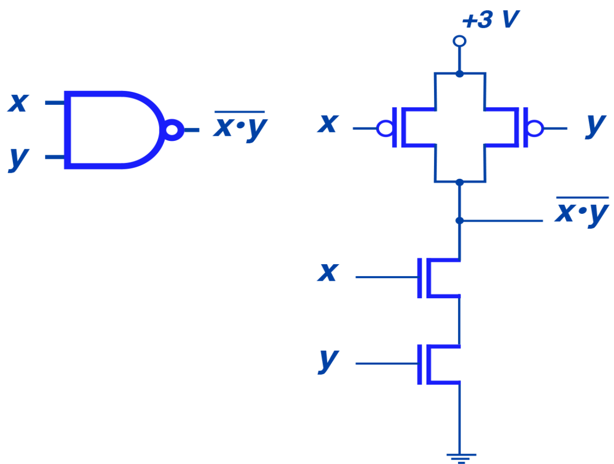


- **NAND**, **NOR** and **NOT** gates can be constructed from two to four transistors.

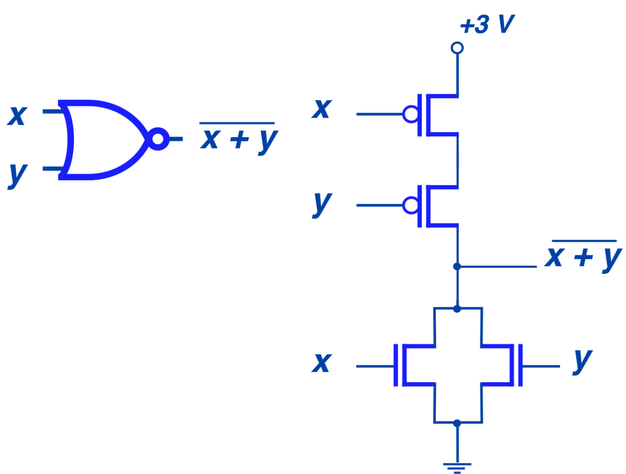
- **NOT**



- **NAND**

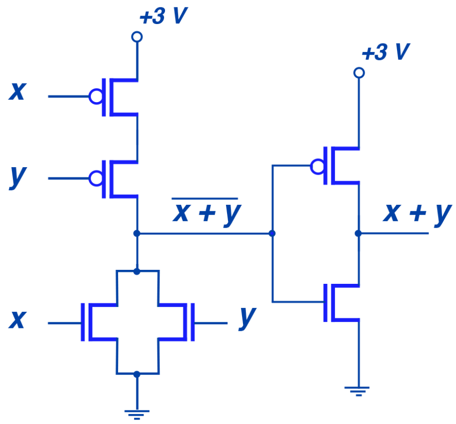


- **NOR**



- **AND** and **OR** gates
  - Require at least six **CMOS** transistors.

- *Example:* OR gate = NOR plus NOT

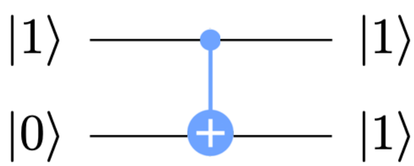


- Integrated circuit layout for **NOT**, **NAND** and **NOR** gates, using CMOS.
- Zoom down inside an IC to see gates!

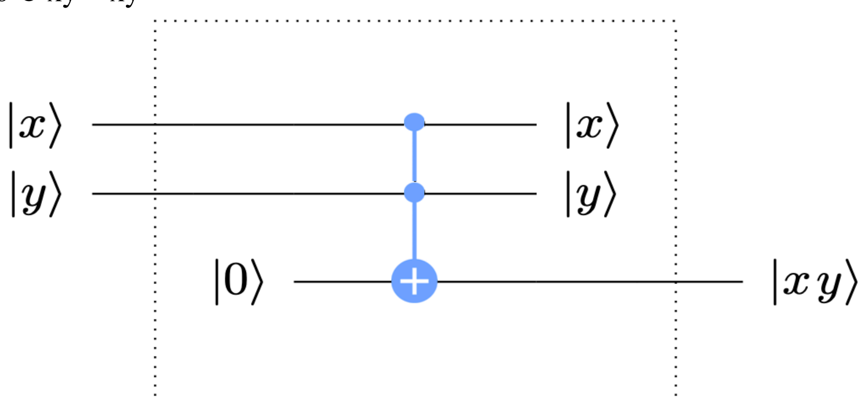
Quantum Computing Implementation of Logic Gates

- **Pauli X (NOT) gate**
  |0> State becomes |1> State

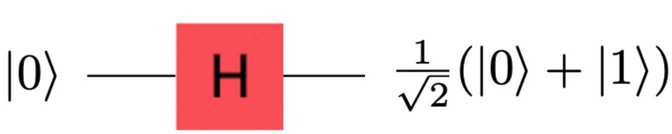
- **CNOT gate** (using Toffoli)
  |0> State becomes |1> State if the Control is |1>



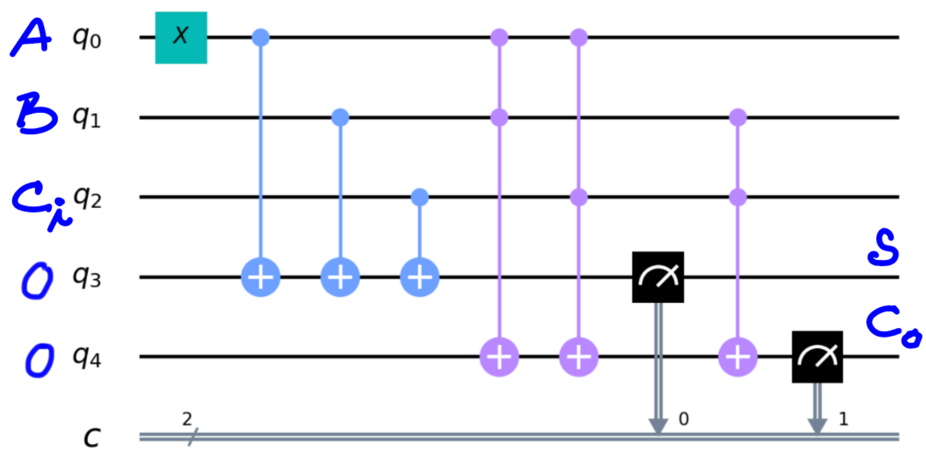
- **AND gate** (using CCNOT Toffoli)
  $0 \oplus xy = xy$



- **H gate**
  |0> State becomes 50% superimposed with |1> State



- Example Quantum Full Adder Circuit



(Image: thequantuminsider.com)

Quiz #3 & selected solutions

MINTERMS

Suppose we want to expand a certain ***SOP*** function f_1 into canonical form whereby each resulting product term contains a literal of every independent variable of f_1.

$$f_1(A,B,C) = \bar{A}\cdot B + B\cdot\bar{C} + A\cdot B\cdot C = \text{SOP}$$
$$= \bar{A}B(C+\bar{C}) + B\bar{C}(A+\bar{A}) + ABC$$
$$f_1(A,B,C) = \underbrace{\bar{A}B\bar{C} + \bar{A}BC + AB\bar{C} + ABC}_{\text{minterms}} = \text{canonical SOP}$$

The product terms above are called **minterms**, the properties of which are now be presented.

Minterm Properties and Notation

- A **minterm** is a product term which produces a single **1** in a truth table

row	ABC	m_2: $\bar{A}B\bar{C}$ (010)	m_3: $\bar{A}BC$ (011)	m_6: $AB\bar{C}$ (110)	m_7: ABC (111)	f_1	$\bar{f_1}$
0	000	0	0	0	0	0	1
1	001	0	0	0	0	0	1
2	010	1	0	0	0	1	0
3	011	0	1	0	0	1	0
4	100	0	0	0	0	0	1
5	101	0	0	0	0	0	1
6	110	0	0	1	0	1	0
7	111	0	0	0	1	1	0

- The minterm which yields a **1** in row i is denoted as minterm m_i
- Express f in terms of minterms:
  - Compose a minterm list for f. (An atomic list)

- *Example:*

$$f_1 = \bar{A}B\bar{C} + \bar{A}BC + AB\bar{C} + ABC$$
$$= m_2 + m_3 + m_6 + m_7$$
$$= \Sigma m(2,3,6,7)$$

- Additional properties of minterm lists
  - ***f*** *=* ***Σm****(row#s where* ***f*** *= 1)*
  - ***f*** *' =* ***Σm****(row#s where* ***f*** *= 0)*
  - ***Σm****(all row#s) =* **1**
- Minterm index *i*
  - Obtained by determining the row code for which $\boldsymbol{m_i}$ = **1**
  - *Example:*

$\bar{A}B\bar{C} = 1$ where ?

0 1 0 in row 2

2

$\therefore \bar{A}B\bar{C} = m_2$

Quiz #4 & selected solutions (also includes ***Maxterms*** discussed below)

K-MAPS

Karnaugh Maps

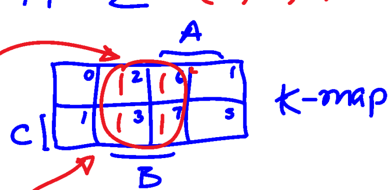
What is a K-map? It is a graphical tool that quickly finds minimal algebraic forms of Boolean functions. The ***SOP*** forms are discussed here; ***POS*** forms are described in a later section.

Karnaugh Map Properties

- Each cell in a K-map for a function *f* corresponds to a row of the truth table describing *f*
- **Cell *i*** is a place mark for minterm $\boldsymbol{m_i}$.
- K-map labels identify the *coincidence of literals* to combine adjacent minterm.

row	f_1	$\bar{f_1}$
0	0	1
1	0	1
2	1	0
3	1	0
4	0	1
5	0	1
6	1	0
7	1	0

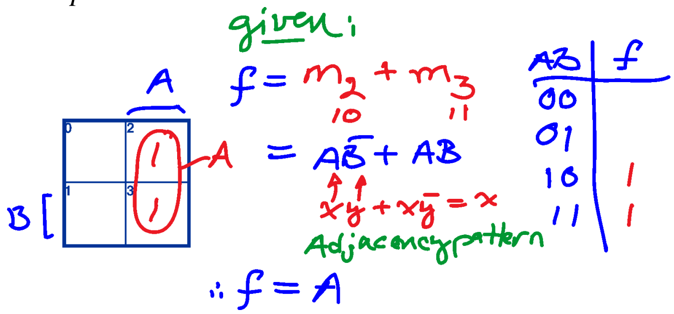
$f_1 = \Sigma m(2,3,6,7)$



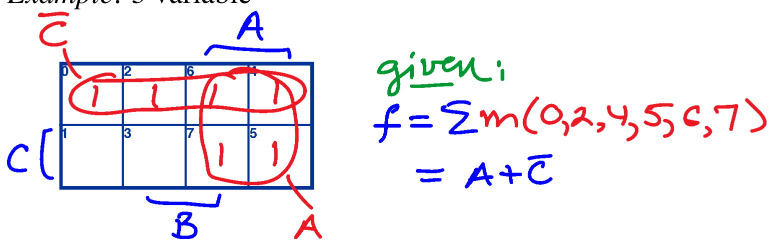
$\boxed{f_1 = B}$

Sizes of K-maps

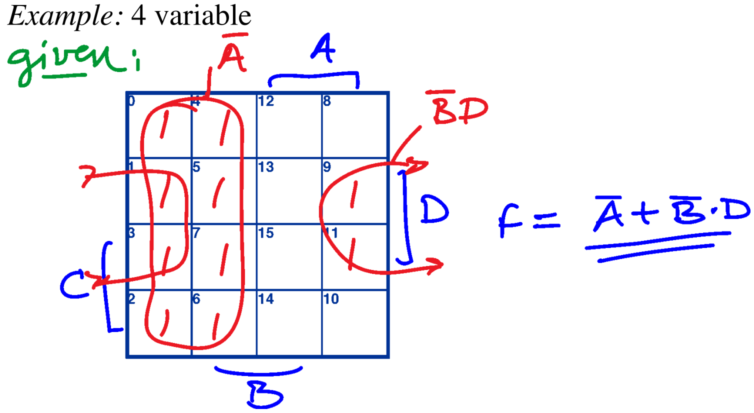
- *Example:* 2 variable



- *Example:* 3 variable

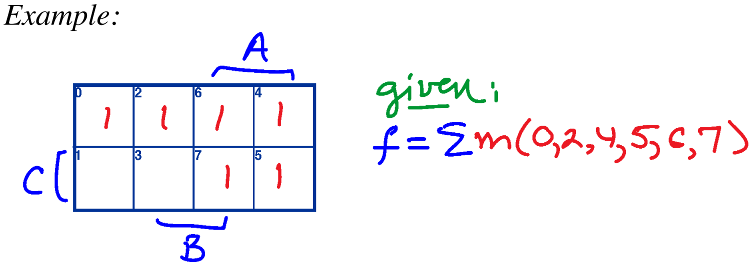


- *Example:* 4 variable



Procedure for Plotting *SOP* Functions on a K-map

- **Determine the minterms m_i** contained in f (found by observing the rows where $f = 1$ in the truth table).
- **Plot the 1's of the function** to be minimized on the K-map.
  - For each minterm m_i in f, enter a **1** in **cell *i*.**
  - *Example:*

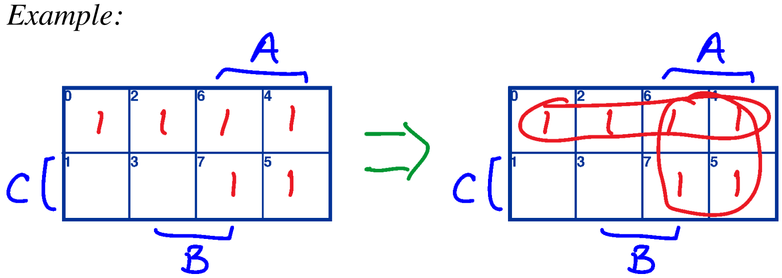


- For each *don't care* contained in f, enter a ***d*** (or ***x***) in the associated K-map cell (see example later).

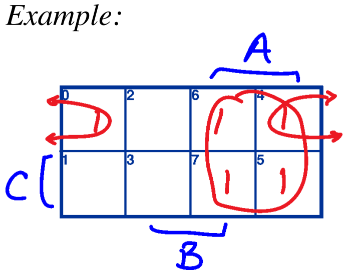
Procedure for *reading* minimal *SOP* expressions from K-maps

- Draw **loops** around **adjacent 1**-entries (cells with **1**'s) in largest groups possible.
  - Group size **in a power of two** (e.g. 1 cell, 2 cells, 4 cells, 8 cells, etc.)

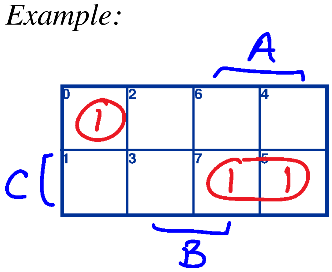
- *Example:*



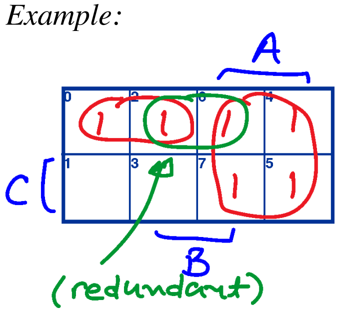
- Cells over the left and right edges, or the upper and lower edges are also *defined* to be adjacent.
  - *Example:*



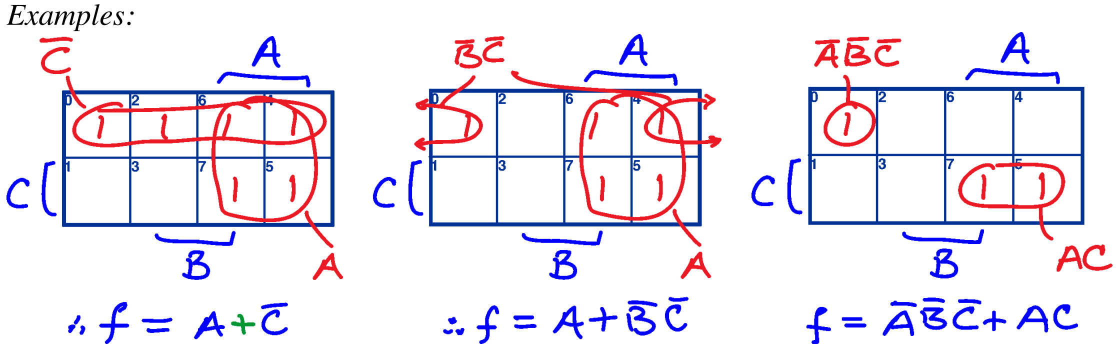
- **1**-entries not adjacent to other **1**-entries are circled as groups of one.
  - *Example:*



- Discard **redundant groupings**
  - Discard those entries covered entirely by other groups
  - *Example:*



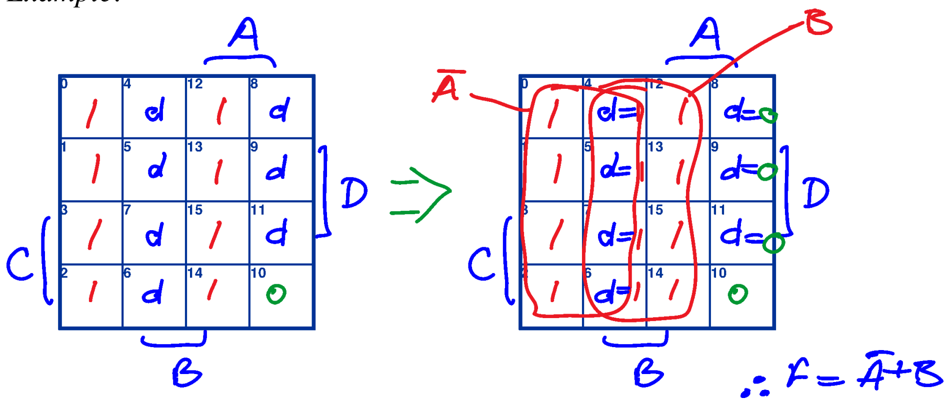
- For each group (as seen in the several examples above),
  - **Read** off the **coincident literals**, by exploiting K-map labels
  - **AND** those literals together to form products formed by the groups
  - **OR** the resulting products to create a SOP expression
  - *Examples:*



Map Simplification Resulting from Don't Cares

- Don’t care = ***d*** (or ***X***) = {0,1} (either a 0 or a 1)
- Group **1**-entries as before, but

- Also include any ***d***-entries which serve to increase the size of the group of **1's**
- Treat unused ***d***-entries as **0**-entries
- Must have at least one **1**-entry in all groups
- *Example:*



- Never group cells consisting **entirely** of don't care entries. This results in a redundant group.

Quiz #5 & selected solutions (includes ***POS*** usages)

MAXTERMS & K-MAPS

Maxterm Properties and Notation

Now we want to consider expanding a certain ***POS*** function f_2 into canonical form whereby each resulting sum term contains a literal of every independent variable of f_2.

product

$$f_2(A,B,C) = \underbrace{(A+B)}_{\text{sum}} \cdot \underbrace{(\bar{A}+B+C)}_{\text{sum}} \cdot \underbrace{(\bar{A}+B+\bar{C})}_{\text{sum}} = POS$$

$$= (A+B+C)\cdot(A+B+\bar{C})\cdot(\bar{A}+B+C)\cdot(\bar{A}+B+\bar{C}) = \text{canonical } POS$$

Maxterms

The sum terms above are called **Maxterms**, the properties of which now follow.

- A **Maxterm** is a sum term which produces a single **0** in a truth table

$n=3$

row	ABC	M_0 $(A+B+C)$	M_1 $(A+B+\bar{C})$	M_4 $(\bar{A}+B+C)$	M_5 $(\bar{A}+B+\bar{C})$	f_2	$\bar{f_2}$
0	000	0	1	1	1	0	1
1	001	1	0	1	1	0	1
2	010	1	1	1	1	1	0
3	011	1	1	1	1	1	0
4	100	1	1	0	1	0	1
5	101	1	1	1	0	0	1
6	110	1	1	1	1	1	0
7	111	1	1	1	1	1	0

- Maxterm which yields a ***0*** in row i is denoted as maxterm M_i
- Expressing ***f*** in terms of Maxterms:
  - Compose a Maxterm list for ***f***. (An atomic list)

- *Example:*

$$f_2 = (A+B+C)\cdot(A+B+\bar{C})\cdot(\bar{A}+B+C)\cdot(\bar{A}+B+\bar{C})$$
$$= M_0 \cdot M_1 \cdot M_4 \cdot M_5$$
$$= \Pi M(0,1,4,5)$$

- Additional properties of Maxterm lists
  - $f = \Pi M$(*row#s where* $f = 0$)
  - $f' = \Pi M$(*row#s where* $f = 1$)
  - ΠM(*all row#s*) = **0**

- Maxterm index i
  - Obtained by determining the row code for which $M_i = \mathbf{1}$
  - *Example:*

$\bar{A}+B+\bar{C} = 0$ where ?

$\downarrow \ \downarrow \ \downarrow$

$\underbrace{1\ 0\ 1}_{5}$ in row 5

$\therefore \bar{A}+B+\bar{C} = M_5$

Other Properties of minterms and Maxterms

- $f = \Sigma m$(*row#s*) $= \Pi M$(*opposite row#s*)
  $f = \Pi M$(*row#s*) $= \Sigma m$(*opposite row#s*)
- If $f = \Sigma m$(*row#s*) then $f' = \Sigma m$(*opposite row#s*)
  If $f = \Pi M$(*row#s*) then $f' = \Pi M$(*opposite row#s*)
  - *Examples:*

For our 2 tables earlier:

$$f_1 = \Sigma m(2,3,6,7) = \Pi M(0,1,4,5) = f_2$$
$$\bar{f_1} = \Sigma m(0,1,4,5) = \Pi M(2,3,6,7) = \bar{f_2}$$

- $m_i' = M_i$
  $M_i' = m_i$
  - *Examples:*

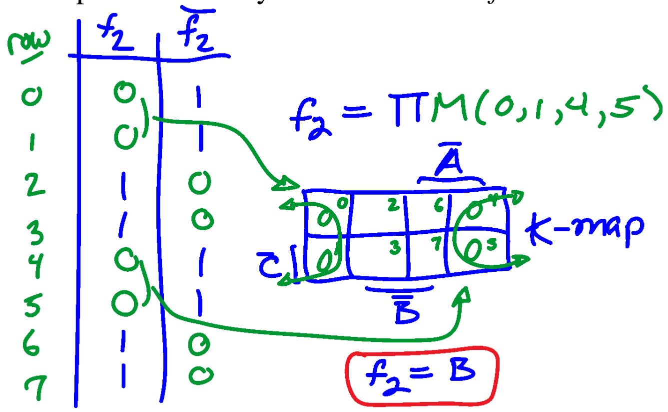
$$m_i + M_i = m_i + \bar{m}_i = \underline{1}$$
$$(x + \bar{x} = 1)$$

$$m_i \cdot \bar{M}_i = m_i \cdot m_i = m_i$$
$$(x \cdot x = x)$$

K-Map *POS* Properties

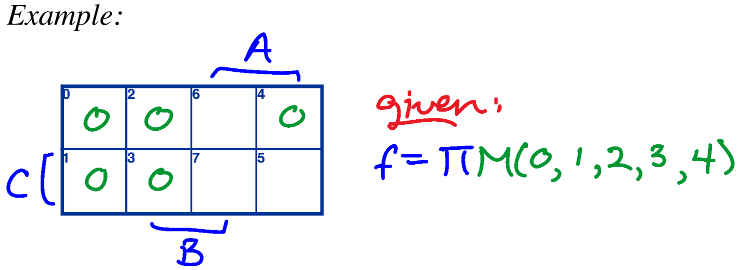
- **Cell** i is a place mark for Maxterm M_i.

- K-map labels identify the *coincidence of literals* to combine adjacent Maxterms.



Procedure for plotting and reading minimal POS expressions from K-maps

- **Plot the 0'*s* of the function** to be minimized on a K-map.
    - For each maxterm M_i in f, enter a **0** in **cell *i*.**
    - *Example:*



- Draw **loops** around **adjacent 0**-entries.
- For each group
    - **Read** off the <u>complement</u> of the **coincident literals** covering the group
    - **OR** those literals together to form sums
    - **AND** the resulting sums to create a product
    - *Example:*